MATH 203 HWK 8

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Question 10

Proof. Let f be a continuous mapping of the compact metric space X into metric space Y. Proceeding by contradiction assume that f is not uniformly continuous. Then for some $\varepsilon > 0$ and for all $\delta > 0$ there exists x, y such that $d_x(x, y) < \delta$ and $d_y(f(x), f(y)) > \varepsilon$.

Then choose sequences $\{p_n\}$ and $\{q_n\}$ such that $d(p_n, q_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} d_x(p_n, q_n) = 0$$

but,

$$d_y(f(p_n), f(q_n)) > \varepsilon$$

Because f is continuous and X is compact, $f(X)$ is compact. By theorem 2.37, $\{f(p_n)\}$ and $\{f(q_n)\}$ have limit points in $f(X)$, call them $f(p)$ and $f(q)$ respectively. These must be distinct or else there would exists some N such that if $n \geq N$ then

$$\begin{gathered}
d_y(f(p_n), f(p)) + d_y(f(q), f(q_n) = d_y(f(p_n), f(p)) + d_y(f(p), f(q_n) \\
d_y(f(p_n), f(p)) + d_y(f(q), f(q_n) \geq d(f(p_n), f(q_n)) \\
2(\varepsilon/2) > d(f(p_n), f(q_n)) \\
\varepsilon > d(f(p_n), f(q_n))
\end{gathered}$$

Thus,

$$\lim_{n\to\infty} f(p_n) \neq \lim_{n\to\infty} f(q_n)$$

By the first equation,

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} p_n$$

We can apply f to both sides since it is continuous, which yields the following

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} f(q_n)$$

Thus a contradiction is achieved. □

Question 11

Proof. Assume f is a uniformly continuous function from the metric space X to a metric space Y. Let $\{x_n\}$ be a Cauchy sequence in X. Then, for all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that if $m \geq n \geq N$, then

$$d_x(x_m, x_n) < \delta$$

Because f is uniformly continuous, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$d_x(x, y) < \delta$$

Then,

$$d_y(f(x), f(y)) < \varepsilon$$

By the results of $\{x_n\}$ being Cauchy, it follows that for the same N, if $m \geq n \geq N$, then

$$d_y(f(x_m), f(x_n)) < \varepsilon$$

This means that $\{f(x_n)\}$ is a Cauchy sequence.

Proof of Continuity Let E be a dense subset of the metric space X. Let f be a uniformly continuous real function defined on E. Then, for each $x \in X/E$, we can construct a sequence $\{x_n\}$ that converges to x by taking each x_n to be an element of E in the ball $B(x, \frac{1}{n})$.

Then, define

$$f(x) = \lim_{n \to \infty} f(x_n)$$

Because f maps into the real numbers it is also complete, hence $\{f(x_n)\}$ is a Cauchy sequence. Under this definition $f(x)$ is continuous for all X. □

Question 12

Statement: let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous functions. Prove that $g \circ f$ is uniformly continuous.

Proof. By the definition of uniform continuity, for all $\eta > 0$ there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$ if

$$d_x(x_1, x_2) < \delta$$

then,

$$d_y(f(x_1), f(x_2)) < \eta$$

Also, for all $\varepsilon > 0$ there exists some $\eta > 0$ such that for all $y_1, y_2 \in Y$ if

$$d_y(y_1, y_2) < \eta$$

then,

$$d_z(g(y_1), g(y_2)) < \varepsilon$$

Thus, for all $\varepsilon > 0$ there exists an η for which there exists a δ so that for all $x_1, x_2 \in X$, if

$$d_x(x_1, x_2) < \delta$$

then

$$d_z(g(f(x_1)), g(f(x_2))) < \varepsilon$$

Thus the composition is uniformly continuous and the statement holds. □

Question 13

Proof. Let E be a dense subset of the metric space X. Let f be a uniformly continuous real function defined on E. For each positive integer n and $p \in X$, let $V_n(p)$ be the set of all $q \in E$ in the ball $B(p, \frac{1}{n})$.

By construction as n goes to infinity, $\text{diam} V_n(P)$ tends to 0. Hence, for all $\varepsilon > 0$ we can find an N such that if $n \geq N$, then

$$\text{diam } f(V_n(p))$$

Because each of these sets is nested and nonempty, their intersection must consist of one point, $g(p)$.

For any $x \in E$, $g(x) = f(x)$. Hence, if this function is continuous, it is a continuous extension of f.

By the same argument as the proof of continuity in question 11, this function is continuous. □

Question 14

Proof. Let $g(x) = x - f(x)$. Because f maps into $[0, 1]$, there exists some $a \in [0, 1]$ such that $f(a) = 0$ and $b \in [0, 1]$ such that $f(b) = 1$. Thus,

$$g(a) = a$$
$$g(b) = b - 1$$

Hence,

$$g(b) \leq 0 \leq g(a)$$

Because f and x are continuous, so is g. Thus, by theorem 4.23 there exists some $x \in [a, b]$ such that $g(x) = 0$ So there exists some x such that

$$0 = x - f(x)$$
$$x = f(x)$$

Since $x \in [a, b]$, $x \in [0, 1]$. Thus the proof is completed. □

Question 15

Proof. Proceeding by contradiction, assume $f : X \to Y$ is open and continuous. Assume it is not monotonic. Then, for some $a, b \in X$ there exists a $c \in (a, b)$ such that $f(a) < f(c)$ and $f(b) < f(c)$. Then, there must exists some $z \in (a, b)$ for which $f(z)$ is a maximum.

Thus the maximum of the image is contained in the open interval and hence the image of the open set (a, b) after applying f cannot be open. □

Question 16

Proof. For any interval that does not contain an integer, each of these functions are continuous as there exists some constant c such that

$$[x] = c$$
$$(x) = x - c$$

which are each clearly continuous.

Take c to be an integer. For any $x \in (c - 1, c)$, $[x] = c - 1$. Hence, for any $\{t_n\} \to x$ such that $t_n < c$

$$[c-] = \lim_{n\to\infty} [t_n] = c - 1$$

On the other hand, for any $x \in (c, c+1)$, $[x] = c$. Hence, for any $\{s_n\} \to x$ such that $s_n > c$

$$[c+] = \lim_{n\to\infty} [s_n] = c$$

Since the limits are not equal, there is a simple discontinuity at every integer.

As for (x), it shares the same discontinuities as it is a combination of the continuous function x and $[x]$. □

Question 17

Proof. Let f be a real function defined on (a, b). Let E be the set of points x such that $f(x-) < f(x+)$. Let p be a rational number such that,

$$f(x-) < p < f(x+)$$

Then, for any rational $\{q_n\} \in (a, x)$ which converges to x, $f(q_n) \to f(x-)$. Hence, there exists an N such that for all $n > N$

$$d(f(q_n), f(x-)) < d(p, f(x-))$$

Let $q = q_N$. Then for any $q < t < x$, $f(t) < p$.

Using the same reasoning for $f(x+)$ choose r such that if $x < t < r$, then $f(t) > p$.

Because both q and r are elements of rational sequences they themselves are rational. Hence, the triple (p, q, r) is made up of rationals which are each from countable sets. Thus the set of these triples are themselves countable.

Assume that this triple (p, q, r) could be associated with another point of E called y. Without loss of generality assume $y > x$. Then

$$f(y-) < p < f(y+)$$

or else our assumption is violated. The other characterizations of these numbers must hold true, namely that for any $q < t < y$

$$f(t) < p$$

However, there exists some $t < r$ such that

$$q < x < t < y$$

Because $x < t < r$,

$$f(t) > p$$

Hence we have shown that $f(t) > p$ and $f(t) < p$ which is a contradiction. So, this triple can only be associated with one element of E.

With this we know that the E is at most equal to the set of triples which are themselves countable. Hence E is at most countable. The same argument works for the set of discontinuities where $f(x-) > f(x+)$. Combining these sets results in another at most countable set. □

Question 18

Proof. Consider the function f defined on R^1 where m and n are coprime integers.

$$f(x) = \begin{cases} 0 & \text{(x is irrational)} \\ \frac{1}{n} & \left(x = \frac{m}{n}\right) \end{cases}$$

Let x be irrational. Let $\varepsilon > 0$. Let N be the closest integer greater than $\frac{1}{\varepsilon}$. Choose

$$\delta = \min\left\{|1-x|, |\frac{1}{2}-x|, |\frac{1}{3}-x|, |\frac{2}{3}-x|, \ldots, |\frac{N-1}{N}-x|\right\}$$

Then, for any rational y such that $|y-x| < \delta$, with $y = \frac{m}{n}$, $n > N$. Hence,

$$|f(y)| = |\frac{1}{n}| < |\frac{1}{N}| < |\varepsilon|$$

Also, if y is irrational,

$$|f(y)| = 0 < \varepsilon$$

Thus $\lim_{n\to\infty} = 9$ so f is continuous at each of the irrationals as $f(x) = 0$.

Now consider $x \in \mathbb{Q}$. Then $x = \frac{n}{m}$ for some coprime integers m and n. Hence, $f(x) = \frac{1}{n}$. However, this is not the limit of f at the point. Because the limit exists, this is a simple discontinuity. □