MATH 203 HWK 7

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Question 2

Proof. Let $F : X \to Y$ be a continuous mapping between two metric spaces. Let $E \subset X$. Take $x \in \overline{E}$. Then, for all $\delta > 0$ there exists a $y \in E$ such that $y \neq x$ and $d_X(x, y) < \delta$.

Because f is continuous at x, for all $\varepsilon > 0$ there exists a $\delta > 0$ so for any $z \in E$ if $d_X(x, z) < \delta$, then,

$$d_Y(f(z), f(x)) < \varepsilon$$

Hence, there exists y so that $d_X(y, x) < \delta$ and hence $d_Y(f(y), f(x)) < \varepsilon$ for any positive ε.

By design, $y \in E$. Because f is continuous,

$$f(y) \in f(E)$$

Hence, $f(x) \in \overline{f(E)}$. Since x is an arbitrary element of $\overline{E}$ it follows that,

$$f(\overline{E}) \subset \overline{f(E)}$$

□

Question 3

Proof. Let $f : X \to \mathbb{R}$ be a real valued, continuous function from a metric space X. Let $Z(f)$ be the set of values p such that,

$$f(p) = 0$$

To show this set is closed, consider the compliment, $Z^c(f)$ which has elements q so that,

$$f(q) \neq 0$$

We want to show that this set is open for any choice of q.

Because f is continuous, if $\varepsilon = |f(q)|$ there exists a $\delta > 0$ so that for all $x \in X$ if $d(x,q) < \delta$, then,

$$\begin{gathered}|f(q) - f(x)| < \varepsilon = |f(q)| \\ ||f(q)| - |f(x)|| < |f(q)| \\ |f(q)| - |f(x)| < |f(q)| \\ 0 < |f(x)|\end{gathered}$$

Hence, the ball $B(q,\delta) \subset Z^c(f)$ as for each $x \in B(q,\delta)$

$$\begin{gathered}0 < |f(x)| \\ f(x) \neq 0\end{gathered}$$

Thus $Z^c(f)$ is open and so $Z(f)$ is closed. $\square$

Question 4

Proof. Let f and g be continuous mappings from a metric space X onto a metric space Y. Let E be a dense subset of X. Hence every point $x \in X$ is a limit point of E. Because f is continuous, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ if $d(y,x) < \delta$ then,

$$d(f(x), f(y)) < \varepsilon$$

Because E is dense in X no matter what δ, we can find a element $e \in E$ such that $d(e,x) < \delta$. Hence,

$$d(f(e), f(x)) < \varepsilon$$

Hence, for any $\varepsilon > 0$ and $x \in X$, we can find an element $f(E)$ that isn't $f(x)$ contained within the ball $B(f(x),\varepsilon)$. Thus $f(E)$ is dense in $f(X)$.

Assume $g(p) = f(p)$ for all $p \in E$. Define $h(x) : X \to Y$ as

$$h(x) = g(x) - f(x)$$

Then, for any p,

$$h(p) = 0$$

Also, if $h(p) = 0$, then,

$$g(p) = h(p)$$

So, $Z(h) = E$. By problem 3, E is closed. Because E is closed and dense, $E = X$. Hence, $p \in x$ and

$$g(p) = f(p)$$

$\square$

Question 5

Proof. Let f be a real continuous function on a closed set $E \subset \mathbb{R}^1$. Since E is closed, E^c is open and because we are working in $\mathbb{R}^1$, it can be expressed as a series of open intervals (a_n, b_n) including possibly the intervals $(-\infty, a)$ and (b, ∞). Construct g as follows,

$$g(x) = \begin{cases} f(x) & \text{if } x \in E \\ f(a_n) + \frac{(x-a_n)}{(b_n-a_n)}(f(b_n) - f(a_n)) & \text{if } x \in (a_n, b_n) \\ f(a) & \text{if } x \in (\infty, a) \\ f(b) & \text{if } x \in (b, \infty) \end{cases}$$

Now to prove continuity, we must take each case individually.

Case 1 Assume $x < a$. This proof also works for $x > b$ by reversing inequalities. Let $\varepsilon > 0$. Let $\delta = a - x$. Then for any $y \in X$ if

$$\begin{gathered} |y - x| < a - x \\ y - x < a - x \\ y < a \end{gathered}$$

Thus, $g(y) = f(a)$. By assumption, $g(x) = f(a)$. So

$$|g(x) - g(y)| = 0 < \varepsilon$$

Case 2 Assume $x \in (a_n, b_n)$. Let $\varepsilon > 0$. Choose $\delta = \min\{x - a_n, b_n - x, \frac{(b_n-a_n)\varepsilon}{|f(b_n)-f(a_n)|}\}$. Then, for all $y \in X$, if

$$\begin{gathered} |y - x| < \delta \\ |y - x| < x - a_n \\ y > a_k \\ \text{and} \\ |y - x| < b_n - x \\ y < b_n \end{gathered}$$

Thus, $y \in (a_n, b_n)$. Also,

$$\begin{gathered} |y - x| < \frac{(b_n - a_n)\varepsilon}{|f(b_n) - f(a_n)|} \\ \frac{|y - x||f(b_n) - f(a_n)|}{b_n - a_n} < \varepsilon \\ |g(y) - g(x)| < \varepsilon \end{gathered}$$

Hence, the function g is continuous on these intervals.

Case 3 Assume $x \in E$. If $x \in E^o$, then there exists a δ_1 so that if $d(x,y) < \delta_1$) then, $y \in E$. Because f is continuous, for each $\varepsilon > 0$ there exists a δ_2 so that for all $y \in X$, if $d(x,y) < \delta_2$ then

$$|f(x) - f(y)| < \varepsilon$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then for any point y, if $d(x,y) < \delta$ then $y \in E$ and hence,

$$|g(x) - g(y)| = |f(x) - f(y)| < \varepsilon$$

Now consider the case that $x \notin E^o$. Then x is one of the endpoints of the closed intervals. Let $\varepsilon > 0$. Because f is continuous, there exists a $\delta_1 > 0$ such that for any point y within δ_1 of x on the E side,

$$|f(x) - f(y)| < \varepsilon$$
$$|g(x) - g(y)| < \varepsilon$$

Now to look at the non-E side. Let $\delta_2 = \frac{(b_n - a_n)\varepsilon}{|f(b_n) - f(a_n)|}$. Then for any y on the non-E side if $d(x,y) < \delta_2$

$$\frac{|y - x||f(b_n) - f(a_n)|}{b_n - a_n} < \varepsilon$$
$$|g(y) - g(x)| < \varepsilon$$

Hence, with $\delta = \min\{\delta_1, \delta_2\}$ the conditions of continuity are met at these points.

□

Question 6

Proof. Suppose E is compact. Let f be a function defined on E.

(=>) Suppose that f is continuous on E. Let $\{x_n\}$ be a sequence in E. This also defines a sequence in the graph, $(x_n, f(x_n))$. Because E is compact, it is also sequentially compact and so there exists a subsequence x_{n_k} and value $x \in E$ such that

$$\lim_{k \to \infty} x_{n_k} = x$$

Because f is continuous, we can apply it to this statement. Thus,

$$\lim_{k \to \infty} f(x_{n_k}) = f(x)$$

Now consider the sequence in the graph $(x_n, f(x_n))$. By the argument above, there exists a subsequence $(x_{n_k}, f(x_{n_k}))$ such that

$$\lim_{k \to \infty} (x_{n_k}, f(x_{n_k})) = (x, f(x))$$

Hence, this arbitrarily chosen sequence in the graph of f has a convergent subsequence. Thus, the graph of f is sequentially compact and consequently compact.

(<=) Assume the graph of f is compact and thus sequentially compact. Proceeding by contradiction assume f is not continuous on E. Then f must not be continuous at a point, call it $x \in E$. Then there exists a $\varepsilon > 0$ such that for all $\delta > 0$ there exists a point $y \in E$ for which $d(x, y) < \delta$ and

$$d(f(x), f(y)) \geq \varepsilon$$

Choose y_n so that $d(x, y_n) < \frac{1}{n}$ and $d(f(x), f(y_n)) \geq \varepsilon$ for $n \in \mathbb{N}$. This defines a sequence $\{(y_n, f(y_n))\}$ in the graph of f. By the sequential compactness of f this sequence has a convergent subsequence, call it $\{(y_{n_k}, f(y_{n_k}))\}$.

Because $d(x, y_{n_k}) < \frac{1}{n}$,

$$\lim_{k\to\infty} y_{n_k} = x$$

Also, because $d(f(x), f(y_{n_k})) \geq \varepsilon$, no subsequence of $\{y_n\}$ converges to $f(x)$. Thus, no subsequence of $\{(y_n, f(y_n))\}$ converges to $(x, f(x))$.

Since $\{y_n\}$ converges to x, and f only takes the value of $f(x)$ at x, $f(y_n)$ must converge to $f(x)$ for $\{(y_n, f(y_n))\}$ to converge to an element of the graph of f. Because this isn't true, no subsequence of $(y_n, f(y_n))$ converges to an element of the graph of f. This contradicts the assumption that any sequence of the graph of f has a subsequence that converges to an element in the graph of f. Hence, the graph of f isn't sequentially compact, which is a contradiction. Thus, f must be continuous. $\square$

Question 7

Proof. Define f and g on $\mathbb{R}^2$ as follows,

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = 0 \\ \frac{xy^2}{(x^2+y^4)} & \text{if } (x, y) \neq 0 \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) = 0 \\ \frac{xy^2}{(x^2+y^6)} & \text{if } (x, y) \neq 0 \end{cases}$$

We want to find the radius a bound for $f(x, y)$, observe the following

$$0 \leq (x - y^2)^2 = x^2 - 2xy^2 + y^4$$
$$\frac{2xy^2}{x^2 + y^4} \leq 1$$
$$\frac{2xy^2}{x^2 + y^4} \leq \frac{1}{2}$$

Hence, f is bounded for all values in $\mathbb{R}^2$. Now consider g. Let $\varepsilon > 0$, let $(y^3, y) \in B((0, 0), \varepsilon)$ and by contradiction assume that $M \geq \max\{g(y^3, y), 1\}$.

Choose the point $(\frac{y^3}{M^3}, \frac{y}{M})$. It is clear that this point lies within the epsilon ball as $|\frac{y^3}{M^3}| < |y^3|$ and $|\frac{y}{M}| < |y|$. The value at this point is the following

$$g(\frac{y^3}{M^3}, \frac{y}{M}) = \frac{\frac{y^3}{M^3}(\frac{y}{M})^2}{((\frac{y^3}{M^3})^2 + (\frac{y}{M})^6))}$$
$$g(\frac{y^3}{M^3}, \frac{y}{M}) = \frac{M}{y}$$

The ball around 0 always contains points with $y < 1$, hence there is always a point inside the ball for which,

$$|g(\frac{y^3}{M^3}, \frac{y}{M})| > M$$

Thus, g is unbounded in any ball around $(0, 0)$.

To show that f is not continuous at $(0, 0)$ consider the following limit,

$$\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^4}{2y^4}$$
$$\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{1}{2}$$
$$\lim_{y\to 0} f(y^2, y) = \frac{1}{2}$$

Hence, f is not continuous at $(0, 0)$.

Now consider the restrictions of f and g to lines. When the lines don't pass through $(0, 0)$ it is clear that there are no discontinuities. As such we need only consider lines that do pass through $(0, 0)$. To do this, let $y = mx + b$ for some constants m and b.

$$f(x, mx) = \frac{x(mx)^2}{(x^2 + (mx)^4)}$$
$$f(x, mx) = \frac{m^2x^3}{(x^2(1 + m^2x^2))}$$
$$\lim_{x\to 0} f(x, mx) = \frac{mx}{1 + m^2x^2}$$
$$\lim_{x\to 0} f(x, mx) = 0$$

So, these lines are continuous at $(0, 0)$ and thus are continuous on $\mathbb{R}^2$. □

Question 8

Proof. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Then choose a $\varepsilon > 0$, by the uniform continuity of f there exists a $\delta > 0$

such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

E can be written as the union of some interval $\{E_n\}$. Because E is bounded and nonempty, each E_n is bounded and nonempty. So, $\sup(E_n)$ and $\inf(E_n)$ exist. Let $k \in \mathbb{N}$. For each of these interval divide then into delta sized chunks as follows,

$$\{E_{n_k}\} = [\inf(E_n) + (k-1)\delta, \inf(E_n) + \min\{\frac{3k}{2}\delta, \sup(E_n)]$$

Then each $\{E_{n_k}\}$ is closed and bounded and hence compact. Thus, by theorem 4.15, f is bounded on $\cup E_{n_k}$ and $E \subset \cup E_{n_k}$. So, f is bounded on E.

As a counter example, take $f(x) = x$ over $\mathbb{R}^1$. This function is uniformly continuous as for $\delta = \varepsilon$, if $|x - y| < \delta = \varepsilon$

$$|x - y| < \varepsilon$$

But also, it is clearly unbounded at by our selection of domain, x is unbounded. □

Question 9

Proof. Assume $f : X \to Y$ is uniformly continuous. Then, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X$, $y \in X$ and $d(x, y) < \delta$,

$$d(f(x), f(y)) < \varepsilon$$

Choose z such that $d(z, x) = d(z, y) = \frac{1}{2}d(x, y)$. Then, let $E = B(z, \frac{1}{2}d(x, y)$. Then diam $E = d(x, y)$. By the given, if diam $E < \delta$ then,

$$\text{diam } f(E) < \varepsilon$$

As each possible combination of $p, q \in E$ lies within δ of each other and hence each $d(f(p), f(q)) < \varepsilon$. Which leads to the conclusion above. □

Part (B)

Proof. Suppose $f : (0, 1] \to \mathbb{R}$ is continuous.

(=>) Assume f is uniformly continuous. By question 8 of the homework, f is bounded. In order to construct the extension, call it g, we just need to let

$$g(0) = \lim_{x \to 0} f(x)$$

Let $\varepsilon > 0$. Hence, we want to show that for all $0 < x < \delta$, there exists some $g(0)$ so that

$$|g(0) - f(x)| < \varepsilon$$

We know that there exists a δ so that for any x, y with $|x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$. Then, if $|x| < \delta$ there exists some $g(0)$ such that,

$$|g(0) - f(y)| < \varepsilon$$

Define $g : [0, 1] \to \mathbb{R}$ as follows,

$$g(x) = \begin{cases} f(x) & \text{if } x \in (0, 1] \\ g(0) & \text{if } x = 0 \end{cases}$$

(<=) Assume f has a continuous extension, g on $[0, 1]$. Then, g is uniformly continuous on $[0, 1]$ as it is a compact set. Then,

$$g((0, 1]) = f((0, 1])$$

Because g is uniformly continuous on this interval, f is as well. $\square$