MATH 203 HWK 6

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Question 7

Proof. Let $\{a_n\}$ be a sequence such that $a_n > 0$. Assume $\sum a_n$ converges. By the Cauchy Shwarz Inequality,

$$\sum \frac{\sqrt{a_n}}{n} \leq \left(\sum a_n\right)^{1/2} \left(\sum 1/n^2\right)^{1/2}$$

$\sum a_n$ converges and by the p-test since $2 > 1$, $\sum 1/n^2$ also converges. Hence, the term on the right is some real number and thus the sequence of partial sums is bounded. That sequence is monotone as $a_n > 0$. By the monotone convergence theorem, the sequence of partial sums converges and hence so does the series itself. $\square$

Question 8

Proof. Suppose that $\sum a_n$ converges and that $\{b_n\}$ is monotonic and bounded. By the monotone convergence theorem $\{b_n\}$ converges to some number, call it b. Then the sequence $\{b_n - b\} = \{c_n\}$ converges to 0 monotonically. This means either

$$c_0 \geq c_1 \geq c_2 \geq \dots$$
or
$$c_0 \leq c_1 \leq c_2 \leq \dots$$

By selecting $\{b - b_n\}$ in the second case we can ensure that the top inequality holds. Also, the partial sums of $\sum a_n$ clearly are bounded as they converge.

By theorem 3.42 $\sum a_n c_n$ converges. Hence either,

$$\sum a_n c_n = \sum a_n(b_n - b) = \sum a_n b_n - \sum a_n b$$
or
$$\sum a_n c_n = \sum a_n(b - b_n) = \sum a_n b - \sum a_n b_n$$

In either case, this implies that $\sum a_n b_n$ converges. $\square$

Question 9

(a)

Proof. We will employ the ratio test to find the radius of convergence. Then examine the following limit,

$$\lim_{n\to\infty} \frac{(n+1)^3 z^{n+1}}{n^3 z^n} = z \lim_{n\to\infty} (\frac{1}{n} + 1)^3 = z$$

Because these limits exist, they are equivalent to the lim sup. So, if $|z| < 1$ the series converges and if $|z| > 1$ it diverges. Thus, $R = 1$. □

(b)

Proof. Once more utilize the ratio test.

$$\lim_{n\to\infty} \frac{2^{n+1} z^{n+1} n!}{(n+1)! 2^n z^n} = \lim_{n\to\infty} \frac{2z}{n+1} = 0$$

Because these limits exist, they are equivalent to the lim sup. So, for any z the series converges. Thus, $R = \infty$. □

(c)

Proof. Examine the following limit in accordance with the ratio test

$$\lim_{n\to\infty} \frac{2^{n+1} z^{n+1} n^2}{(n+1)^2 2^n z^n} = \lim_{n\to\infty} \frac{2z}{(\frac{1}{n} + 1)^2} = 2z$$

Because these limits exist, they are equivalent to the lim sup. So, if $|2z| < 1$ the series converges and if $|2z| > 1$ it diverges. Thus, $R = 1/2$. □

(d)

Proof. Examine the following limit in accordance with the ratio test

$$\lim_{n\to\infty} \frac{(n+1)^3 z^{n+1} 3^n}{3^{n+1} n^3 z^n} = \lim_{n\to\infty} \frac{(\frac{1}{n} + 1)^3 z}{3} = \frac{z}{3}$$

Because these limits exist, they are equivalent to the lim sup. So, if $|\frac{z}{3}| < 1$ the series converges and if $|\frac{z}{3}| > 1$ it diverges. Thus, $R = 3$. □

Question 10

Proof. Let $\{a_n\}$ be a sequence of integers such that there are an infinite number of elements distinct from 0. Then, for any $N \in \mathbb{N}$ there exists an $n \geq N$ so that $a_n \neq 0$ so $d(a_n, 0) \geq 1$. This contradicts the statement that $\{a_n\}$ converges to 0.

Now for the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}z^{n+1}}{a_n z^n} = \lim_{n\to\infty} \frac{a_{n+1}z}{a_n} = z \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Suppose $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$. Because $a_n \in \mathbb{Z}$, $|a_{n+1}| \leq |a_n| - 1$. Hence, when $N = a_0$, for each $n \geq N$, $a_n = 0$. This means $\{a_n\}$ converges to 0 which contradicts the assumption. Hence $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| \geq 1$ and so the radius of convergence must be less than or equal to 1. $\square$

Question 11

Suppose $a_n > 0$, $s_n = a_0 + a_1 + ... + a_b$, and $\sum a_n$ diverges.

(a)

Proof. s_n is monotone because $a_n > 0$. Because it doesn't converge, it must be unbounded. Assume that $\lim\limits_{n\to\infty} \frac{a_n}{1+a_n} = 0$. Then, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ so that if $n \geq N$ then,

$$d(\frac{a_n}{1+a_n}, 0) < \varepsilon$$

Hence,

$$|\frac{a_n}{1+a_n}| \leq |\frac{a_N}{1+a_N}| < \varepsilon$$
$$a_n < \varepsilon + \varepsilon a_N = M_1$$

Then we can take the $M_2 = \max\{a_1, a_2, \ldots, a_N\}$. Then $M = \max\{M_1, M_2\}$ bounds $\{a_n\}$ which is a contradiction.

Thus, $\lim\limits_{n\to\infty} \frac{a_n}{1+a_n} \neq 0$ so $\sum \frac{a_n}{1+a_n}$ diverges. $\square$

(b)

Proof. We will begin by combining terms for the right side of the inequality. Let $k \in \mathbb{N}$.

$$1 - \frac{s_N}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}}$$
$$1 - \frac{s_N}{s_{N+k}} = \frac{a_{N+1} + a_{N+2} + \cdots + a_{N+k}}{s_{N+k}}$$

Because $\{s_n\}$ is monotonically increasing as $a_n > 0$,

$$\frac{a_{N+1} + a_{N+2} + \cdots + a_{N+k}}{s_{N+k}} \leq \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}}$$

We assume that $\{s_n\} \to \infty$. Hence, $\lim_{k\to\infty} \frac{s_N}{s_{N+k}} = 0$. Consequently,

$$\lim_{k\to\infty} 1 - \frac{s_N}{s_{N+k}} = 1$$

So, there exists a K for which

$$1 - \frac{s_N}{s_{N+K}} = \frac{1}{2} \leq \sum_{n=N+1}^{N+K} \frac{a_n}{s_n}$$

Then, for each $M \in \mathbb{N}$, we can divide the sum into K length sums as follows. Let m be the closest integer greater than M/K.

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n} = \sum_{i=0}^{\infty} \sum_{n=iK+1}^{iK+K} \frac{a_n}{s_n}$$

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n} \geq \sum_{i=0}^{\infty} \frac{1}{2}$$

The right hand diverges as the limit of $1/2$ is not 0. Hence, by the squeeze theorem $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges as well. □

(c)

Proof. We will attempt to prove the following inequality;

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

First, simplify the RHS

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n}$$

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{a_n}{s_n s_{n-1}}$$

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} \geq \frac{a_n}{s_n^2}$$

The final step is possible because $s_{n-1} \leq s_n$. Next to show that $\sum \frac{a_n}{s_n^2}$ converges.

$$\sum_{n=1}^{N} \frac{a_n}{s_n^2} = \frac{a_1}{s_1^2} + \sum_{n=2}^{N} a_n s_n^2$$

$$\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

Notice that the series on the RHS telescopes leaving only the first term and last term. Hence,

$$\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \frac{1}{a_1} + \frac{1}{s_1} - \frac{1}{s_N}$$

Because $a_n > 0$,

$$0 < \sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \frac{1}{a_1} + \frac{1}{s_1} - \frac{1}{s_N}$$

Thus, taking the limits as N tends to infinity yields the following,

$$0 < \sum_{n=1}^{\infty} \frac{a_n}{s_n^2} \leq \frac{1}{a_1} + \frac{1}{s_1}$$

Hence, the series converges, particularly to a value within these bounds. □

(d)

Proof. First consider the series

$$\sum \frac{a_n}{1 + n^2 a_n}$$

We can rewrite the sequence to produce the following inequality,

$$\frac{a_n}{1 + n^2 a_n} = \frac{1}{\frac{1}{a_n} + n^2} \leq \frac{1}{n^2}$$

Hence,

$$0 \leq \sum \frac{a_n}{1 + n^2 a_n} \leq \sum \frac{1}{n^2}$$

By the p-series test, the RHS converges which implies that

$$\sum \frac{a_n}{1 + n^2 a_n}$$

does as well.

Now consider the following series,

$$\sum \frac{a_n}{1 + n a_n}$$

We cannot use the same procedure as above since $\sum \frac{1}{n}$ diverges. If $\{a_n\} = \frac{1}{n}$, then

$$\sum \frac{a_n}{1 + n a_n} = \sum \frac{1/n}{1 + 1}$$
$$\sum \frac{a_n}{1 + n a_n} = \frac{1}{2} \sum \frac{1}{n}$$

Hence, the series diverges. However, if $\{a_n\} = \frac{1}{n^2}$,

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1/n^2}{1+\frac{1}{n}}$$

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1}{n^2+n} \leq \sum \frac{1}{n^2}$$

Then, by the p-series test the RHS converges. Hence, by the comparison test, our series also diverges. Consequently, we cannot say for certain whether this series converges or diverges for arbitrary $\{a_n\}$. $\square$

Question 12

Suppose $a_n > 0$ and $\sum a_n$ converges. Put,

$$r_n = \sum_{m=n}^{\infty} a_m$$

(a)

Proof. We will attempt to prove the following inequality. Let $m < n$

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

Begin with the left hand side,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m}$$

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{r_m - r_n}{r_m}$$

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

Thus,

$$\sum_{i=m}^{n} \frac{a_i}{r_i} > 1 - \frac{r_n}{r_m}$$

Because $n > m$, $r_n > r_m$. We can choose an N large enough so that

$$\frac{r_N}{r_m} < \frac{1}{2}$$

So,

$$\sum_{i=m}^{N} \frac{a_i}{r_i} > \frac{1}{2}$$

Let p be the desired width of the intervals so that $p = N - m$. Let N_j be the next multiple of p greater than N. Take some j so that $jp = N_j$. Then we can split the sum into a sum of sums.

$$\sum_{i=1}^{N_j} \frac{a_i}{r_i} = \sum_{i=0}^{j-1} \sum_{n=ip+1}^{ip+p} \frac{a_n}{r_n}$$

$$\sum_{i=1}^{N_j} \frac{a_i}{r_i} > \sum_{i=0}^{j-1} 1 - \frac{r_{ip+p}}{r_{ip+1}}$$

$$\sum_{i=1}^{N_j} \frac{a_i}{r_i} > \sum_{i=0}^{j-1} \frac{1}{2}$$

As N_j tends to infinity, so does j, consequently,

$$\sum_{i=1}^{\infty} \frac{a_i}{r_i} > \sum_{i=0}^{\infty} \frac{1}{2}$$

So, the left hand side diverges. $\square$

(b)

Proof. First let us try and prove the following inequality,

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

Rewrite the left hand side,

$$\frac{a_n}{\sqrt{r_n}} = \frac{r_n - r_{n+1}}{\sqrt{r_n}}$$

$$\frac{a_n}{\sqrt{r_n}} = \sqrt{r_n} - \frac{r_{n+1}}{\sqrt{r_n}}$$

$$\frac{a_n}{\sqrt{r_n}} = \sqrt{r_n} - \frac{r_{n+1}}{\sqrt{a_n + r_{n+1}}}$$

$$\frac{a_n}{\sqrt{r_n}} < \sqrt{r_n} - \frac{r_{n+1}}{\sqrt{r_{n+1}}}$$

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

Now to show the series of the left side converges. Consider the finite sums up to some $N \in \mathbb{N}$.

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < \sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

By observation, the right hand side is telescoping so we can rewrite it as follows.

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\sqrt{r_1} - 2\sqrt{r_{N+1}}$$

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\sqrt{\sum_{n=1}^{\infty} a_n} - 2\sqrt{\sum_{n=N+1}^{\infty} a_n}$$

Because $a_n > 0$,

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\left(\sum_{n=1}^{\infty} a_n - \sum_{n=N+1}^{\infty} a_n\right)$$

$$0 < \sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\left(\sum_{n=1}^{N+1} a_n\right)$$

As N tends to infinity, the far right side tends towards a finite limit and hence the middle is squeezed, implying that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges. $\square$