MATH 203 HWK 4

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Question 1

(=>)

Proof. Let E be a disconnected set. Then there exists A and B such that $E = A \cup B$ and

$$\overline{A} \cap B = \varnothing$$
$$A \cap \overline{B} = \varnothing$$

Because $A \subset \overline{A}$, $A \cap B = \varnothing$. If A or B is empty, then $E = A$ or $E = B$ which contradicts the assumption that E is disconnected. Hence A and B are nonempty.

By contradiction assume A is not open relative to E. Then for all $\epsilon > 0$ and $a \in A$, $B(a, \epsilon)$ contains a point $b \in E \cap A^c = E \cap B$. Then $a \in \overline{B}$ which is a contradiction since $A \cap \overline{B} = \varnothing$.

Identically, by contradiction assume B is not open relative to E. Then for all $\epsilon > 0$ and $b \in B$, $B(b, \epsilon)$ contains a point $a \in E \cap B^c = E \cap A$. Then $b \in \overline{A}$ which is a contradiction since $B \cap \overline{A} = \varnothing$. □

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Proof. Let $E = A \cup B$ where A and B nonempty, $A \cup B = \varnothing$ and A and B are both open relative too E.

We want to show that $\overline{A} \cap B = \varnothing$. By contradiction, take $x \in \overline{A} \cap B$. Because $A \subset \overline{A}$, $x \in A$. There exists an $\epsilon > 0$, because B is open relative to E, so that $B(x, \epsilon) \cap E \subset B$. This produces a contradiction as $x \in A$ and $x \in B(x, \epsilon)$ yet $A \cup B = \varnothing$. Hence $\overline{A} \cap B = \varnothing$.

Identically, we want to show that $\overline{B} \cap A = \varnothing$. By contradiction, take $x \in \overline{B} \cap A$. Because $B \subset \overline{B}$, $x \in B$. There exists an $\epsilon > 0$, because A is open relative to E, so that $B(x, \epsilon) \cap E \subset A$. This produces a contradiction as $x \in B$ and

$x \in B(x, \epsilon) \subset B$ yet $A \cup B = \varnothing$. Hence $\overline{B} \cap A = \varnothing$.

It follows that E is disconnected. $\square$

Question 17

Proof. Let E be the set of all $x \in [0, 1]$ such that the decimal expansion of x contains only 4 and 7. For each $x \in E$, define the sequence a_n such that the nth element is 1 if the nth element of the decimal expansion is a 4 and a 0 if it is a 7. Then, let A be the collection of these sequences.

$$A = \{a_n \mid x \in E\}$$

This collection contains every sequence whose elements are 1 and 0. By theorem 2.14, A is uncountable. We have described a bijection between A and E so E is also uncountable.

Take $y = .1$. Let $\epsilon = .1$. Then $B(y, \epsilon) = (0, .2)$. There exists no element of E in this interval because elements must start with either .4 or .7. Hence $E \subset [.4, .8]$. Consequently, E is not dense in $[0, 1]$.

E is bounded as for $x \in E$, $d(0, x) < .8$. It remains to show that E is closed to prove it compact. Take $y \in E^c$. Then the nth decimal of y is neither 4 or 7, lets call this number a. Call the $n + 1$th decimal b. Let the decimal expansion of ϵ be all zeros except for the $n + 1st$ term which is a 1. For $b = [1, 8]$, $B(y, \epsilon)$ is the following interval,

$$(... \frac{a}{10^n} + \frac{b-1}{10^{n+1}} ..., ... \frac{a}{10^n} + \frac{b+1}{10^{n+1}} ...)$$

For these values of b, every $x \in B(y, \epsilon)$ has an a in the nth decimal position, which means $x \in E^c$. Next, consider $b = 9$, then $B(y, \epsilon)$ is the following

$$(... \frac{a}{10^n} + \frac{8}{10^{n+1}} ..., ... \frac{a+1}{10^n} + \frac{0}{10^{n+1}} ...)$$

Even if $a + 1$ might be a 4 or 7, we know that the $n + 1$st term is 0. Hence, all x in this interval are also members of E^c. Lastly, consider $b = 0$ for which the interval takes the form,

$$(... \frac{a-1}{10^n} + \frac{9}{10^{n+1}} ..., ... \frac{a}{10^n} + \frac{1}{10^{n+1}} ...)$$

In this case, even if $a - 1$ is a 4 or 7, these numbers have a 9 in the $n + 1$st decimal position meaning they are all members of E^c. Hence, E^c is open which in turn means E is closed. Consequently, E is compact.

Take $0.4 \in E$. Let $\epsilon = 0.01$. Then,

$$B(0.4, 0.01) = (0.39, 0.41)$$

This ball contains no elements of E as all other elements start with 0.7 or 0.44. Hence, for all other $x \in E$, $x \notin B(0.4, 0.01)$. Thus, all elements of E are not limit points so E is not perfect. □

Question 18

Proof. Let $n = 0, 1, 2, \ldots$. Let F_n be defined as follows,

$$F_0 = [0, 1]$$
$$F_1 = [0, 1/3] \cup [2/3, 1]$$
$$F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$
$$\vdots$$

Then the Cantor set is $C = \bigcap_{n=0}^{\infty} F_n$. Because each F_n is closed, C is closed. Each $F_{n+1} \subset F_n$ so by theorem 2.38, C is non-empty. Take $x \in C$. Let $\epsilon > 0$. There exists n sufficiently large that

$$\frac{1}{3^n} < \epsilon$$

Then there are two cases.

Case 1:

Assume x is an upper bound for its interval. Then, for all $n = 0, 1, 2, \ldots$ there exists a point of C at $x - \frac{1}{3^n}$. Hence for any ϵ, $d(x, x - \frac{1}{3^n}) < \epsilon$ so $B(x, \epsilon)$ contains another point in C. Consequently $x \in C'$.

Case 2:

Assume x is a lower bound for its interval.Then, for all $n = 0, 1, 2, \ldots$ there exists a point of C at $x + \frac{1}{3^n}$. Hence for any ϵ, $d(x, x + \frac{1}{3^n}) < \epsilon$ so $B(x, \epsilon)$ contains another point in C. Consequently $x \in C'$.

Thus, C is perfect.

Lastly, consider the linear transformation of C so that it operates on the interval $[\sqrt{2}, 1 + \sqrt{2}]$. As a result, $\sqrt{2}$ is added to every value. Thus, every value becomes irrational as an irrational added to a rational is irrational. Hence, $C + \sqrt{2}$ is irrational, nonempty, and perfect. □

Question 19

(a)

Proof. Let A and B be disjoint closed sets in some metric space X. Because they are closed,

$$A = \overline{A}$$
$$B = \overline{B}$$

Also, since they are disjoint,

$$A \cap B = \varnothing$$
$$\overline{A} \cap B = \varnothing$$
$$A \cap \overline{B} = \varnothing$$

Hence, A and B are separated. $\square$

(b)

Proof. Let A and B be disjoint open sets in some metric space X.

Take $a \in \overline{A}$. For each $\epsilon > 0$, $B(a, \epsilon)$ contains another point $a' \in A$. Hence $B(a, \epsilon) \notin B$. Consequently, $a \notin B$ since B is open.

Identically, take $b \in \overline{B}$. For each $\epsilon > 0$, $B(b, \epsilon)$ contains another point $b' \in B$. Hence $B(b, \epsilon) \notin A$. Consequently, $b \notin A$ since A is open.

So,

$$\overline{A} \cap B = \varnothing$$
$$A \cap \overline{B} = \varnothing$$

which means A and B are separated. $\square$

(c)

Proof. Fix $p \in X$. Let $\delta > 0$. Let

$$A = \{q \mid d(p, q) < \delta\}$$
$$B = \{q \mid d(p, q) > \delta\}$$

So, $A = B(p, \delta)$ so A is open. Take $q \in B$, let $d(p, q) = \delta + \epsilon$ for some $\epsilon > 0$. Let $z \in B(q, \epsilon/2)$, so $d(q, z) < \epsilon/2$. By the reverse triangle inequality,

$$d(p, q) - d(q, z) < d(p, z)$$

We know that $\delta + \epsilon < d(p,q)$ and $d(q,z) < \epsilon/2$. Hence,

$$\delta + \epsilon - \epsilon/2 < d(p,z)$$
$$\delta < d(p,z)$$

So, $z \in B$. Hence B is open. Lastly, by the trichotomy order axiom, if $x \in A$, $d(p,x) < \delta$, $x \notin B$ and vice-versa. Thus A and B are disjoint. By part b, A and B are separated. □

(d)

Proof. Let a and b be elements of a connected metric space. Then, for every $\epsilon \in [0, d(a,b)]$ there needs to exist a point c such that $d(a,c) = \epsilon$ or else by part c the metric space would be separated. Hence, there is bijection between points in our connected metric space and the interval $[0, d(a,b)]$. By the corollary to theorem 2.43, our metric space is uncountable. □

Question 20

Proof. Let $y \in \mathbb{R}^2$. Let $\epsilon > 0$. Consider the following sets,

$$A = \{x \mid d(y,x) \leq \epsilon\}$$
$$B = \{x \mid d(y,x) \geq \epsilon\}$$

The union of these sets is $\mathbb{R}^2$ which is connected. However, the interior of the union is the following,

$$A = \{q \mid d(p,q) < \epsilon\}$$
$$B = \{q \mid d(p,q) > \epsilon\}$$

which we proved in question 19.c to be separated.

Take E to be a connected subset of in a metric space. Let $\overline{E} = A \cup B$ where

$$\overline{A} \cap B = \varnothing$$
$$A \cap \overline{B} = \varnothing$$

Also,

$$E = (E \cap A) \cup (E \cap B)$$

Because $(E \cap A) \subset A$, $\overline{(E \cap A)} \subset \overline{A}$. Also, $\overline{(E \cap B)} \subset \overline{B}$. Hence,

$$\overline{(E \cap A)} \cap B = \varnothing$$
$$A \cap \overline{(E \cap B)} = \varnothing$$

It follows that,

$$\overline{(E \cap A)} \cap (E \cap B) = \varnothing$$
$$(E \cap A) \cap \overline{(E \cap B)} = \varnothing$$

Hence $E \cap A$ and $E \cap B$ are separated and separate E. This is a contradiction to the assumption that E is connected. Hence $\overline{E}$ must be connected. $\square$