MATH 203 HWK 3

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Question 12

Proof. Let $K \subset R^1$ be the set of points $1/n$ for $n \in \mathbb{N}$ and 0. Let $\{G_\alpha \mid \alpha \in \mathbb{A}\}$ be an open cover of K. Then we know that 0 is contained in at least one of these G_α, without loss of generality $0 \in G_0$. Because G_0 is open, there exists and $\epsilon > 0$ such that if $d(0,x) < \epsilon$, then $x \in G_0$. Then there exists an $N \in \mathbb{N}$ such that for each $n > N$, $1/n < \epsilon$. Hence for each $n > N$, the points $1/n \in G_0$. This leaves us the set of finite points $n = 1, 2, ...N$. Since they are in K, they are covered in the open cover and because they are finite, there are a finite $G_{\alpha 1}, G_{\alpha 2}, ...G_{\alpha N}$ which cover them. Hence,

$$K \subset G_0 \cup G_{\alpha 1} G_{\alpha 2} \cup ... \cup G_{\alpha N}$$

Thus there exists a finite subcover. □

Question 14

Proof. Take the cover $\{G_n \mid n \in \mathbb{N}\}$ where

$$G_n = (1/n, 1)$$

Then by the archimedes property, for each real number $x \in (0,1)$ there exists an $n > 1/x$. Hence,

$$0 < 1/n < x < 1$$

Consequently,

$$(0,1) \subset \bigcup_{n \in \mathbb{N}} G_n$$

However, take an arbitrary finite subset $n_1, n_2 ... n_k$. Let $m = max(n_1, n_2, ...n_K)$. Then, by the density of real numbers there exists an x such that,

$$0 < x < 1/m < 1$$

Because m is the max, $1/m < 1/n_k$ for $k = 1, ...K$. So x is not covered by the finite subcover. Hence, there doesn't exist a finite subcover. □

Question 15

Proof. Take the set $K_n = (0, 1/n)$ where $n \in \mathbb{N}$. Then, clearly each set is bounded by 1 and 0. However, $\cap K_n = \varnothing$ as for $0 < x < 1$, there exists an n large enough by the archimedean property such that $0 < 1/n < x < 1$ so that $x \notin K_n$. Hence $x \notin \cap K_n$.

Take the set $K_n = [n, \infty)$. Take an arbitrary limit point x of K_n, for all $\epsilon > 0$, there exists another point $y \in K_n$ such that $d(x, y) < \epsilon$. Hence, is closed. However, for any $x \in [n, \infty)$, there exists a $N \in \mathbb{N}$ such that $x < N$. Hence $x \notin K_N$ so $x \notin \cap K_n$. Thus, $\cap K_n = \varnothing$ $\square$

Question 22

Proof. Take the metric space $\mathbb{R}^k$. Consider subset, $\mathbb{Q}^k$ which is the set of points with rational coordinates. Then $\mathbb{Q}^k = \mathbb{Q} \times \mathbb{Q} \times ... \times \mathbb{Q}$. Because $\mathbb{Q}$ is countable, so too is $\mathbb{Q}^k$. Take any two points $\overline{x}, \overline{y} \in \mathbb{Q}^k$. Then we have

$$\overline{x} = (x_1, x_2, ..., x_k)$$
$$\overline{y} = (y_1, y_2, ..., y_k)$$

Without loss of generality assume $x_n < y_n$ for $n = 1, 2, ..., k$. Because each $\mathbb{Q}$ is dense, there exists $\overline{z}$ such that,

$$\begin{gathered} x_1 \leq z_1 \leq y_1 \\ x_2 \leq z_2 \leq y_2 \\ \cdots \\ x_k \leq z_k \leq y_k \end{gathered}$$

Hence,

$$\overline{x} \leq \overline{z} \leq \overline{y}$$

So $\mathbb{Q}^k$ is countable and dense, hence $\mathbb{R}^k$ is separable. $\square$

Question 23

Proof. Let X be a separable metric space. Then there is a countable and dense subset $X' \subset X$. For each $x' \in X'$, consider the set balls with a rational radius, call it Z. Because the rationals are countable this will result in a countable number of points each with countable balls. Hence, the set of these open subsets is countable.

To show that this set is a base, take an arbitrary $x \in X$ and open subset

$G \subset X$ such that $x \in G$. Then for each $y \in G$ there exists an $\epsilon > 0$ for which $B(y, \epsilon) \subset G$. Because X' is dense, $B(y, \epsilon/2)$ contains an element $x' \in X'$. Additionally, because $\mathbb{Q}$ is dense, there exists a rational number q such that $d(x', y) < q < \epsilon/2$. Then, the ball $B(x', q)$ is in Z, contains the point y, and is contained in $B(y, \epsilon)$. $B(y, \epsilon) \subset G$. Hence, for all $x \in X$ and all open subsets $G \subset X$ that contain x, there is a subset of Z for which G is the union of each element of the subset. Thus, Z is a base for X, and as shown above, Z is countable. □

Question 24

Proof. Let X be a metric space in which every infinite subset has a limit point. Choose $x_1 \in X$. Let $\delta > 0$. Choose $x_2 \in X$ such that

$$d(x_1, x_2) > \delta$$

Continue choosing $x_j \in X$ such that $d(x_i, x_j) > \delta$ for $i = 1, ..., j-1$. This infinite set does not have a limit since the ball $B(x_n, \delta)$ contains no other point for $n = 1, ...j$. This contradicts the assumption of the space, hence, there must be a number N points that can be spaced as required in the space X. Consequently the finite set of balls

$$C_N = \{B(x_1, \delta), B(x_2, \delta), ..., B(x_N, \delta)\}$$

covers X. This is because there are no more points y such that $d(x_k, y) > \delta$ for $k = 1, ..., N$.

Now, let $\delta = 1/n$. To show that C is dense take an arbitrary point $z \in X$, for any $\epsilon > 0$, we can select n sufficiently large so that $\epsilon > \delta > 0$ so that $B(z, \epsilon)$ contains a point of C_N. This is because for every point z in X, there exists a point $c \in C$ such that, $d(z, c) < \delta < \epsilon$. Hence, $c \in B(z, \epsilon)$. Then, the collection of C_n is countable because it is a countable collection of finite sets. Consequently, X is separable. □

Question 25

Proof. Let K be a compact metric space. Then given any open cover of K, we can produce a finite subcover. Let $\epsilon = 1/n$ where n is a natural number. Take the open cover

$$C = \{B(x, \epsilon) \mid x \in K\}$$

Since every ball is open and contains it's own point, it is clearly an open cover of K. Because K is compact for each $n \in \mathbb{N}$, there exists a finite subcover consisting of a finite set of these balls.

Let $\delta > 0$. Take an arbitrary point $k \in K$ and consider the ball $B(k, \delta)$. We can choose n large enough so that $\delta > 1/n > 0$. Let S_n be the set of centers for the balls generated by this particular n. Then for any point k, the ball $B(k, 1/n)$ contains a point of S. It follows that the ball $B(k, \delta)$ also contains that same point since $\delta > 1/n$. We can take the union of these points $\underset{n}{\cup} S_n$ for $n = 1, 2, ...$ and the resulting set will be countable, since each S_n is finite. Additionally, now for any $\delta > 0$, $B(k, \delta)$ contains an element of S, no matter the size of δ. So the union is dense in K. Consequently K is separable by this union. □

Question 26

Proof. Let X be a metric space in which every infinite subset has a limit point. By exercise 24, X is separable. Then for every open cover, there exists a countable subcover $\{G_\alpha\}$, for $\alpha = 1, 2, 3, \ldots$

If there is a finite subcollection of $\{G_\alpha\}$ that covers X, then X is compact. Otherwise, let F_n be the complement of $G_1 \cup \ldots \cup G_n$ for each n. F_n must be nonempty if this subcollection of $\{G_\alpha\}$ doesn't cover. However, $\bigcap F_n$ is empty.

Consider the set E which contains a point from every F_n. Since E is an infinite subset, it must have a limit point. Then there exists some $x \in E$ and for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point of E which is also a point of $\bigcap F_n$. Also, $x \in G_N$ for some α because it is a cover of X. So there is some $\delta > 0$ for which $B(x, \delta) \subset G_N$. However, there exists $\delta > \epsilon > 0$ so that there exists a point from F_N in $B(x, \delta)$ which is a contradiction.

Thus, there must be a finite subcollection of $\{G_\alpha\}$ that covers X. So X is compact.

□

Question 27

Proof. Let $E \subset \mathbb{R}^k$ be an uncountable subset. Let P be all the condensation points of E. Let $\{V_n\}$ be a countable base for $\mathbb{R}^k$ and let $W = \bigcup V_n$ for n such that $V_n \cap E$ is at most countable. Then W is at most countable.

Let $x \notin W$. Because R^k is separable, for all $\epsilon > 0$, x is contained in some V_k such that $x \in V_k \in B(x, \epsilon)$. Because $x \notin W$, $V_k \cap E$ is uncountable. Then each $B(x, \epsilon)$ contains some V_k that has uncountable elements of E. Hence, x is a condensation point, so $x \in P$.

Take $x \notin P$, then there exists ball around x that contains at most countable elements of E. Because $\mathbb{R}^k$ is separable, there exists a V_k contained in this ball.

Then $V_k \cap E$ is at most countable. Hence, $x \in W$. It follows that $P = W^c$.

We have shown that $W \cap E$ is at most countable, since W is a countable union of at most countable sets. Hence $P^c \cap E$ is countable.

Because W is the union of open sets, W is open and so P is closed.

$$P' \subset P$$

Let $x \in P$ and consider an arbitrary ball B. By contradiction assume the ball B contains no other point in P. Then, for all $y \in B$, $y \in W$. Then, $B \cap E \subset W \cap E \cup \{x\}$. Thus, because $W \cap E$ is at most countable, $B \cap E$ is at most countable, which is a contadiction. So B must contain another point of P other than x. So x is a limit point. Hence,

$$P \subset P'$$

It follows that P is perfect $\square$

Question 28

Proof. Let E be a closed set in the separable metric space X. Because E is closed, it contains all of its limit points and condensation points. Let P be the condensation points of E. Then

$$E = P \cup \{E \backslash P\}$$

P is perfect and $E \backslash P$ is at most countable by Question 27. $\square$

Question 29

Proof. Because $\mathbb{R}^k$ is separable, so too is $\mathbb{R}^1$. Consequently, it has a countable base $\{G_\alpha\}$. Let E be an open set in $\mathbb{R}^1$. Because E is open, E is equal to the union of some subcollection of $\{G_\alpha\}$. Call this subscollection B.

$$\cup B = E$$

Becuase $\{G_\alpha\}$ is countable, so it B.

Because we are working in $\mathbb{R}^1$, open sets take the form,

$$(a, b)$$

Then, for every G_x and G_y in B, if

$$G_x \cap G_y \neq \varnothing$$

we can combine them into one open interval. Let $G_x = (x_1, x_2)$ and $G_y = (y_1, y_2)$. Without loss of generality, assume $x_1 \leq y_1$ and $x_2 \leq y_2$. Then we can combine them into one open interval that covers the same set as follows,

$$G_{xy} = (x_1, y_2)$$

The selection of x_1 and y_2 specifically are not important. They need only be the lower of the two lower bounds and the upper of the two upper bounds. With this process,

$$G_x \cup G_y = G_{xy}$$

This process can be carried out for every overlapping G_x and G_y. Then, let the set of remaining open intervals be called A. Since combining two intervals removes one from the set, and B is countable, A is at most countable. Also,

$$\cup A = \cup B$$

So,

$$\cup A = E$$

We have ensured that every $G_x \in A$ is disjoint from any other $G_y \in A$ so, E is equal to an at most countable union of disjoint segments.

$\square$