MATH 203 HWK 2

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A

(i)

Let a, b, c, d be real numbers that define two open sets (a, b) and (c, d). Without loss of generality assume $a < b$ and $c < d$. Define a one-to-one correspondence for $x \in (a, b)$ as follows:

$$f : (a, b) \rightarrow (c, d)$$
$$x \rightarrow x\frac{(d-c)}{(b-a)} + \frac{(bc-ad)}{(b-a)}$$

Because this is a linear function is is clear that an inverse exists, hence it defines a one-to-one correspondence.

(ii)

The same function defined in part i works for closed intervals.

(iii)

Because the rational numbers are countable, let the sequence $x_1, x_2, x_3, ...$ be the rational numbers in the interval (a, b)

$$f : [a, b] \rightarrow (c, d)$$
$$x \rightarrow \begin{cases} x_{n+2}\frac{(d-c)}{(b-a)} + \frac{(bc-ad)}{(b-a)} & \text{if } x = x_n \\ x_1\frac{(d-c)}{(b-a)} + \frac{(bc-ad)}{(b-a)} & \text{if } x = a \\ x_2\frac{(d-c)}{(b-a)} + \frac{(bc-ad)}{(b-a)} & \text{if } x = b \\ x\frac{(d-c)}{(b-a)} + \frac{(bc-ad)}{(b-a)} & \text{if } x \notin \mathbb{Q} \end{cases}$$

Once more, as these are a collection of linear functions, inverses can be produced for each case to recover the initial value. As such it is a one-to-one correspondence.

(iv)

Once more, let the sequence $x_1, x_2, x_3, ...$ be the rational numbers in the interval $(0, 1)$. Then the following function describes a one-to-one correspondence with $\mathbb{R}$.

$$f : [0,1] \to \mathbb{R}$$

$$x \to \begin{cases} \tan(\frac{\pi}{2}x_{n+2}) & \text{if } x = x_n \\ \tan(\frac{\pi}{2}x_1) & \text{if } x = 0 \\ \tan(\frac{\pi}{2}x_2) & \text{if } x = 1 \\ \tan(\frac{\pi}{2}x) & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function describes a one-to-one correspondence as using the function $\frac{2}{\pi}\arctan(x)$ will recover the initial value.

B

(2)

Let $a_0, ..., a_n$ be a sequence of n integers, not all zero. Then let an algebraic number be some $z \in \mathbb{C}$ such that,

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z^1 + a_n = 0$$

Hence we can represent each algebraic number by the unique tuple of coefficients that satisfy the equation above.

$$Z_n = \{(a_0, a_1, a_2 ... a_n) \in \mathbb{Z} \mid a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z^1 + a_n = 0\} \tag{1}$$

This set is a subset of the set of all integer tuples of length n. Because these numbers are taken from the integers, a countable set, by Theorem 2.13 in Rudin, the set Z_n is countable. Then consider the union of the sets Z_n for $n \in \mathbb{N}$.

$$Z = \bigcup_{n \in \mathbb{N}} Z_n$$

Each Z_n is countable, so by Theorem 2.12, Z itself is countable. We can exchange the tuples for the algebraic numbers they represent meaning that the algebraic numbers are countable.

(3)

Assume that every real number is an algebraic number. Then there exists a injective map from real numbers to algebraic numbers. Since the real numbers are uncountable, the image of the map must also be uncountable. However, the image of the mapping is a subset of the algebraic numbers which have been proven to be countable. This is a contradiction. Hence, there exist real numbers that are not algebraic.

(5)

Consider the function

$$f(n) = \begin{cases} \frac{1}{n} & \text{if } n \bmod 3 = 0 \\ \frac{1}{n} + 1 & \text{if } n \bmod 3 = 1 \\ \frac{1}{n} + 2 & \text{if } n \bmod 3 = 2 \end{cases}$$

For all $\epsilon > 0$, by the Archimedean property we can choose an n sufficiently large so that

$$n > \frac{1}{\epsilon}$$

It follows that,

$$\frac{1}{n} < \epsilon$$
$$\frac{1}{n} + 1 < 1 + \epsilon$$
$$\frac{1}{n} + 2 < 2 + \epsilon$$

Without loss of generality assume $n \bmod 3 = 0$. Then we can take

$$\frac{1}{n+1} + 1 < 1 + \epsilon$$
$$\frac{1}{n+2} + 2 < 2 + \epsilon$$

to satisfy the conditions for convergence to 1 and 2 respectively as $(n+1) \bmod 3 = 1$ and $(n+2) \bmod 3 = 2$.

(6)

Let E' be the set of all limit points of a set E. Take E'^c. If $x \in E'^c$, then there exists an $\epsilon > 0$ such that $B(x, \epsilon)$ does not contain a point of E. Hence, $B(x, \epsilon) \subset E^c$. Take $y \in B(x, \epsilon)$ and let $\delta = \epsilon - d(x, y)$. Take $z \in B(y, \delta)$, by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$
$$d(x, z) < d(x, y) + \epsilon - d(x, y)$$
$$d(x, z) < \epsilon$$

Then, for each y, $B(y, \delta) \subset B(x, \epsilon) \subset E^c$. Then each y is a limit point for E^c. Hence, $B(x, \epsilon) \subset E'^c$. Consequently, E'^c is open, so E' is closed.

Take a $x \in E'$, then $x \in \overline{E}$ because $\overline{E} = E \bigcup E'$. Hence, $E' \subset \overline{E}$. Take

$y \in \overline{E}'$, then for all $\epsilon > 0$, $B(y, \epsilon)$ contains a point z in $\overline{E}$. Then by the definition of $\overline{E}$, for all $\delta > 0$, $B(z, \delta)$ contains a point w in E. In particular, let $\delta < \epsilon - d(y, z)$. Then, by the triangle inequality,

$$\begin{aligned} d(y, w) &\leq d(y, z) + d(z, w) \\ d(y, w) &< d(y, z) + \epsilon - d(y, z) \\ d(y, w) &< \epsilon. \end{aligned}$$

So for all $\epsilon > 0$, $B(y, \epsilon)$ contains a point, w, in E. Thus, $y \in E'$.

E and E' do not always share the same limit points. For example take the set $X = \{1/n \mid n \in \mathbb{N}\}$ over the real numbers. Then, for any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. So 0 is the only limit point for X. However, since $X' = \{0\}$, it has no limit points.

(7)

(a)

Take a point $x \in \overline{B_n}$. Then, for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point $y \in B_n$. Since $y \in B_n$, $y \in \bigcup_{t=1}^{n} A_t$. This means that $x \in \bigcup_{t=1}^{n} \overline{A_t}$.

Take $x \in \bigcup_{t=1}^{n} \overline{A_t}$. Then, for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point $y \in \bigcup_{t=1}^{n} A_t$. Hence, $y \in B_n$. So, $x \in \overline{B}$.

(b)

Take a point $x \in \bigcup_{t=1}^{\infty} \overline{A_t}$. Then, for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point $y \in \bigcup_{t=1}^{\infty} A_t$. So, $y \in B$. Thus, $x \in \overline{B}$. Consequently, $\bigcup_{t=1}^{\infty} \overline{A_t} \subset \overline{B}$.

(8)

Let E be an open set in $\mathbb{R}^2$. Take an arbitrary point $x \in E$. Then, there exists an $\delta > 0$, such that $B(x, \delta) \subset E$. For any $\epsilon > \delta$, $B(x, \delta) \subset B(x, \epsilon)$. Hence, $B(x, \epsilon)$ contains a point $q \in E$. Also, for each $0 < \epsilon \leq \delta$, each point $q \in B(x, \epsilon)$ is also in $B(x, \delta)$. This means that $q \in E$. Hence, for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point $q \in E$. Hence, each point in E is a limit point.

Now consider E as a closed set in $\mathbb{R}^2$. Take the set $E = \{(0, 0)\}$ over the field $\mathbb{R}$. This set is closed since there are no limit points. But also, $(0, 0)$ is a not a limit point. Hence there exists a point that is not a limit point.

(9)

(a)

Proof. Let $x \in E^o$. Then, there exists an $\gamma > 0$ such that $B(x, \gamma) \subset E$. Take $y \in B(x, \gamma)$. Let $\delta = \gamma - d(x, y)$. Note that $\gamma > d(x, y)$ because $y \in B(x, \gamma)$. Hence $\delta > 0$. Let $\epsilon = d(x, y)$.

Then for each $z \in B(x, \epsilon)$, we want to show that $B(z, \delta) \subset B(x, \gamma)$. This will show that each $z \in E^o$, which will mean that for all x in E^o, there exists an epsilon such that $B(x, \epsilon) \subset E^o$. Hence, E^o is open.

Let $w \in B(z, \delta)$. We want to show $d(x, w) < \gamma$ as this means $B(z, \delta) \subset B(x, \gamma)$. We know that for each z, $d(x, z) < \epsilon = d(x, y)$. Also we know that $d(z, w) < \delta = \gamma - d(x, y)$. By the triangle inequality,

$$d(x, w) \leq d(x, z) + d(z, w)$$
$$d(x, w) < d(x, y) + \gamma - d(x, y)$$
$$d(x, w) < \gamma$$

Hence, each $w \in B(x, \gamma)$. Consequently, for each $z \in B(x, \epsilon)$, there exists a $\delta > 0$ such that $B(z, \delta) \subset B(x, \gamma)$. So, each $z \in E^o$ which means that for each $x \in E^o$, there exists a $\epsilon > 0$ such that $B(x, \epsilon) \subset E^o$. This means that E^o is open. □

(b)

Proof. (=>)
Assume that E is open. Then for each point $x \in E$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset E$. Hence, $x \in E^o$ as well. Thus $E^o = E$.

(<=)
Assume $E^o = E$. Then for every point x in E and there exists an $\epsilon > 0$, such that $B(x, \epsilon) \subset E$. Hence, E is open. □

(c)

Assume $G \subset E$ and that G is open. Then for all $x \in G$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset G$. Hence, $B(x, \epsilon) \subset E$. Consequently, $x \in E^o$ for each $x \in G$. So $G \subset E^o$.

(d)

Take $x \in (E^o)^c$. Then for all $\epsilon > 0$, $B(x, \epsilon)$ contains a point of E^c. That means that $x \in \overline{E^c}$

Take $y \in \overline{E^c}$. Then for all $\epsilon > 0$, $B(y, \epsilon)$ contains a point of E^c. Hence, $y \in (E^o)^c$.

(e)

Take the set $[-1, 0) \cup (0, 1]$ over the real numbers. Then the closure of this set is $[-1, 1]$. For $1 > \epsilon > 0$, $B(0, \epsilon)$ is contained within $[-1, 1]$. Hence, 0 is in the interior of the closure. However, 0 is not in the initial set and thus cannot be in the interior.

(f)

Take $E = \{0\}$ over the real numbers. Then E^o is empty. So $\overline{E^o}$ is empty. However, $\overline{E} = E \cup E'$. Hence, $E \subset \overline{E}$ so $\overline{E}$ is nonempty.

C

Proof. Let X be the real line with the metric

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Let $a \in X$. Then, $B(a, 1/2) = a$ because the only point $x \in X$ for which $d(x, a) < 1/2$ is a. The same holds for $B(a, 1)$ as for a point x to be a member of this ball, it must satisfy $d(x, a) < 1$. But the only point that satisfies that is a once more.

The closure of $B(a, 1)$ is the union of all points and limit points of $B(a, 1)$. However, the only limit point of $B(a, 1)$ is a. Let b be any point that isn't a. Then let $\epsilon = 1/2$. $B(b, \epsilon) = b$ which doesn't include a. Hence, no other point can be a limit point for $B(a, 1)$. Consequently, the closure of $B(a, 1)$ is a. □