MATH 203 HWK 1

James Gillbrand

October 2024

Question A

Proof. Let

$$S = \left\{ \frac{2n-2}{3n} : n \in \mathbb{N} \right\}$$

S is non-empty. Take $n = 1$, then $0 \in S$. Let $\epsilon > 0$. Because n is a natural number, $n > 0$. Hence,

$$\frac{2}{3} > \frac{2}{3} - \frac{2}{3n} = \frac{2n-2}{3n}$$

Thus, 2/3 is an upper bound for the set S. Also, S is nonempty. Also, by the Archimedean property of natural numbers, for each ϵ, there exists some $n \in \mathbb{N}$ such that

$$\frac{3n}{2} > \frac{1}{\epsilon}$$

Consequently,

$$\begin{aligned} \frac{2}{3n} &< \epsilon \\ \frac{2}{3} - \frac{2}{3n} &> \frac{2}{3} - \epsilon \\ \frac{2n-2}{3n} &> \frac{2}{3} - \epsilon \end{aligned}$$

Hence, 2/3 is the supremum of S. □

Question B

Proof. Let S and T be two bounded, nonemtpy sets of real numbers. Let $s \in S$ and $t \in T$. Then for any s and t, by the

$$\max(\text{sup S, sup T}) \geq \text{sup S} \geq s$$
$$\max(\text{sup S, sup T}) \geq \text{sup T} \geq t$$

So, for $x \in S \cup T$

$$\max(\text{sup S}, \text{sup T}) \geq x$$

For each $\epsilon > 0$, there exist some t and s such that,

$$\text{sup S} - \epsilon \leq s$$
$$\text{sup T} - \epsilon \leq t$$

Without loss of generality, sup S $\geq$ sup T. Then,

$$\max(\text{sup S}, \text{sup T}) = \text{sup S}$$

Hence, for each $\epsilon > 0$, there exists some $s \in S$, such that,

$$\max(\text{sup S}, \text{sup T}) - \epsilon \leq s$$

Because $s \in S \cup T$,

$$\max(\text{sup S}, \text{sup T}) = \text{sup } S \cup T$$

□

Question C

Proof. Let S and T be two bounded sets and let $s \in S$ and $t \in T$. Then, sup S + sup T $\geq s + t$ for any s and t because,

$$\text{sup S} \geq s$$
$$\text{sup T} \geq t$$

Hence, sup S + sup T is an upper bound for the set $S + T$. For each $\epsilon/2 > 0$, there exists some elements s and t such that

$$\text{sup S} - \frac{\epsilon}{2} \leq s$$
$$\text{sup T} - \frac{\epsilon}{2} \leq t$$

Consequently,

$$(\text{sup T} - \frac{\epsilon}{2}) + (\text{sup S} - \frac{\epsilon}{2}) \leq s + t$$
$$\text{sup T} + \text{sup S} - \epsilon \leq s + t$$

Thus,

$$\text{sup T} + \text{sup S} = \text{sup(T+S)}$$

□

Question 1

Proof. Let r be a nonzero rational number and let x be irrational. Then let $p, q, n, m \in \mathbb{Z}$.

$$r = \frac{p}{q}$$

By contradiction assume $r + x = \frac{n}{m}$ and $rx = \frac{n}{m}$. Then,

$$r + x = \frac{p}{q} + x = \frac{n}{m}$$
$$x = \frac{nq - pm}{qm}$$

Because each of p, q, n, m are integers, the fraction is a rational number. Hence a contradiction is achieved. Also,

$$rx = \frac{px}{q} = \frac{n}{m}$$
$$x = \frac{nq}{mp}$$

This fraction is also a rational number, contradicting the assumption. □

Question 2

Proof. Proceeding by contradiction assume that 12 has a rational square root. Then, there exists we can write $12 = p/q$ where p and q are coprime integers. Hence,

$$\left(\frac{p}{q}\right)^2 = 12$$
$$\left(\frac{p}{2q}\right)^2 = 3$$

So it follows that $\sqrt{3}$ is rational. Let $2q = k$, another integer. Hence,

$$p^2 = 3k^2$$

Since p^2 is a multiple of 3, because 3 is prime, it follows that p is a multiple of 3. So we can say $p = 3j$. So,

$$(3j)^2 = 3k^2$$
$$3j^2 = k^2$$

By the same reasoning, k is also a multiple of three. Consequently, q is also a multiple of three. This contradicts the assumption that p and q are coprime. □

Question 5

Proof. Let $\alpha = \text{Sup}(-A)$. Then $-\alpha = -\text{ Sup}(-A)$. By the definition of the least upper bound, for each $\epsilon > 0$, there exists some $x \in A$ such that

$$\alpha - \epsilon < -x$$

Hence,

$$-\alpha + \epsilon > x$$

This satisfies the definition of the infimum of A. Consequently, $-\text{ Sup}(-A) = \text{Inf }(A)$

$\square$

Question 7

Let $b > 1$, $y > 0$, and $x \in \mathbb{R}$.

(a)

Let n be a positive integer.

$$b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + b^{n-3} + ... + 1) \geq (b-1)(n)$$

(b)

By part a,

$$b - 1 = (b^{1/n})^n - 1$$
$$(b^{1/n})^n - 1 \geq n(b^{1/n} - 1)$$

Hence,

$$b - 1 \geq n(b^{1/n} - 1)$$

(c)

Let $t > 1$ and assume that $n > \frac{(b-1)}{(t-1)}$. By assumption,

$$n(t-1) > (b-1)$$

Then by part b,

$$n(t-1) > n(b^{1/n} - 1)$$
$$t - 1 > b^{1/n} - 1$$
$$t > b^{1/n}$$

(d)

Assume that w is such that $b^w < y$. Let $t = \frac{y}{b^w}$ and apply part c. Because $b^w < y$, $y > 1$. Then, for sufficiently large n,

$$\frac{y}{b^w} > b^{1/n}$$
$$y > b^{1/n+w}$$

(e)

Let $b^w > y$. So $\frac{b^w}{y} > 1$. Let $t = \frac{b^w}{y}$ and let n be sufficiently large. Then apply part c,

$$b^{1/n} < \frac{b^w}{y}$$
$$y < \frac{b^w}{b^{1/n}}$$
$$b^{w-1/n} > y$$

(f)

Let A be the set of all w such that $b^w < y$. A is bounded above by y and for $w = 0$, $b^0 < 1 < y$, so A is nonempty. Thus there exists a Sup A, lets call it x. Proceeding by contradiction, assume $b^x \neq y$.

Case 1

Assume $b^x < y$. In this case, let $w = x + 1/n$ for sufficiently large n. As such, $w \in A$ and

$$b^x < b^{x+1/n} < y$$

Hence, x is not a upper bound for A, which contradicts the assumption.

Case 2

Assume $b^x > y$. Then, there exists an n large enough such that $b^{x-1/n} > y$. By the property of least upper bounds, there also exists an element $w \in A$ such that for each n,

$$b^x > y > b^w > b^{x-1/n} > y$$

Which is a contradiction. Consequently,

$$b^x = y$$

(g)

By contradiction, assume that both x and z are both unique real numbers such that,

$$y = b^x = b^z$$

Without loss of generality assume $z < x$. Then,

$$b^x = b^z = b^{z+(x-z)} = b^z b^{x-z}$$

We know that $x - z > 0$, so $b^z = b^z b^{x-z} > b^z$ which is a contradiction.

Question 8

By contradiction assume that there exists some ordering of the complex numbers. Then, we know that squares of numbers in ordered fields are positive. Hence $i^2 = -1 > 0$. Also, $-1^2 = 1 > 0$. However, $-1 + 1 = 0$. Consequently $-1 < 0$ which is a contradiction. So the complex numbers cannot be made into an ordered field.

Question 11

Let z and w be a complex numbers and let $|w| = 1$. Let $r \geq 0$. Take

$$\frac{z}{|z|} = w$$

Then, let $r = |z|$ so that

$$z = rw$$

This combination of r and w is uniquely defined by z. Assume there is some other $x \in \mathbb{C}$ such that $x = rw$. By substituting from the previous equation we know that $x = z$. Thus, z uniquely defines r and w.

Question 14

Let z be a complex number such that $|z| = 1$. Then,

$$\begin{aligned} |1+z|^2 + |1-z|^2 &= (1+z)(1+\overline{z}) + (1-z)(1-\overline{z}) \\ &= 4 \end{aligned}$$